ÉRETTSÉGI VIZSGA ● 2008. május 6.

MATEMATIKA ANGOL NYELVEN

EMELT SZINTŰ ÍRÁSBELI ÉRETTSÉGI VIZSGA

JAVÍTÁSI-ÉRTÉKELÉSI ÚTMUTATÓ

OKTATÁSI ÉS KULTURÁLIS MINISZTÉRIUM

Important Information

Formal requirements:

1. The papers must be assessed in **pen and of different colour** than the one used by candidates. Errors and flaws should be indicated according to ordinary teaching practice.
2. The first one among the shaded rectangles next to each question contains the maximal score for that question. The **score** given by the examiner should be entered into the other **rectangle**.
3. **In case of correct solutions**, it is enough to enter the maximal score into the corresponding rectangle.
4. In case of faulty or incomplete solutions, please indicate the corresponding **partial scores** within the body of the paper.

Substantial requirements:

1. In case of some problems there are more than one marking schemes given. However, if you happen to come across with some **solution different** from those outlined here, please check the parts equivalent to those in the solution provided here and do your marking accordingly.
2. The scores in this assessment **can be split further**. Remember, however, that the number of points given for any item can be an integer number only.
3. If the answer is correct and the argument is clearly valid then the maximal score can be given even if the actual solution is **less detailed** than that in this booklet.
4. If there is a **calculation error** or any other flaw in the solution, then the score should be deducted for the actual item only where the error has occured. If the candidate is going on working with the faulty intermediate result and the problem has not suffered substantial damage due to the error, the subsequent partial scores should still be given.
5. If there is a **fatal error** within an item (these are separated by double lines in this booklet), then even formally correct steps should not be given any points, whatsoever. However, if the wrong result obtained by the invalid argument is used correctly throughout the subsequent steps, the candidate should be given the maximal score for the remaing parts, unless the problem has been changed essentially due to the error.
6. If an **additional remark** or a **measuring unit** occurs in brackets in this booklet, the solution is complete even if the candidate does not mention it.
7. If there are more than one correct attempts to solve a problem, it is the **one indicated by the candidate that can be marked**.
8. You should **not give any bonus points** (points beyond the maximal score for a solution or for some part of a solution).
9. You **should not reduce** the score for erroneous calculations or steps unless its results are actually used by the candidate in the course of the solution.
10. **There are only 4 questions to be marked out of the 5 ones in part II. of this exam paper.** Hopefully, the candidate has entered the number of the question not to be marked in the square provided for this. Accordingly, this question should not be assessed even if there is some kind of solution contained in the paper. Should there be any ambiguity about the student’s request with respect to the question not to be considered, it is the last one in this problem set, by default, that should not be marked.

I.

1.		
Both the even and odd numbered terms of an arithmetic progression form an arithmetic progression themselves.	1 point	*This point is due if this idea appears in the solution.*
There are 11 odd and 10 even numbered terms, respectively..	1 point	
By the conditions: $\frac{a_1 + a_{21}}{2} \cdot 11 = \frac{a_2 + a_{20}}{2} \cdot 10 + 15$.	2 points	*Writing down the respective sums is worth 1 point and indicating their relation based on the text is the other 1 point.*
Let the common difference of the given progression be *d*. The equation hence obtained is: $\frac{a_1 + a_1 + 20d}{2} \cdot 11 = \frac{a_1 + d + a_1 + 19d}{2} \cdot 10 + 15$.	1 points	
When rearranging one gets $2a_1 + 20d = 30$.	1 points	
Similarly rewriting the other condition of the problem: $a_1 + 19d = 3(a_1 + 8d)$.	2 points	
Rearranging again: $2a_1 + 5d = 0$.	1 point	
The solution of the system is: $a_1 = -5$, $d = 2$.	2 points	
The term to be found is $a_{15} = -5 + 14 \cdot 2 = 23$, This value in fact satisfies the conditions of the problem.	1 point	
Total:	**12 points**	

2. a)		
Denote the number of students from the grades 9-10. by a and their mean score by A. Then the size of the 11-12 group is $100-a$, by condition.	1 point	
$a=1.5\cdot(100-a)$, by condition and thus $a=60$. There are 60 students from the grades 9-10 and 40 ones from the grades 11-12.	2 points	
If B denotes the mean score of the 11-12 group then $1.5A=B$.	1 point	
Using the previous results the mean score is: $100=\frac{60A+40B}{100}=\frac{80B}{100}=\frac{4B}{5}$.	2 points	
Hence $B=125$, therefore the mean score of the 11-12 group is 125.	1 point	
Total:	**7 points**	*These 7 points are still due if the candidate is working with the value of A and he/she is using correct roundings.*

2. b)		
There are $\binom{100}{3}$ $(=161700)$ equiprobable ways to choose 3 out of the 100 students.	1 point	
The number of selections according to the conditions is $\binom{60}{2}\cdot\binom{40}{1}$ $(=70800)$,	1 point	
because the pair and the single student are selected independently.	1 point	
The given probability is hence $P=\frac{\binom{60}{2}\cdot\binom{40}{1}}{\binom{100}{3}}$,	1 point	
that is $P=\frac{70800}{161700}$ (≈ 0.44).	1 point	*This 1 point is still due if the result is given as 44%*
Total:	**5 points**	

3.		
It is neccessay and sufficient for a quadratic to have a double real root its discriminant be zero.	1 point	*This point is due if this idea appears in the solution.*
Therefore, the condition for the discriminant is $16(\sin\alpha+\cos\alpha)^2-4\cdot4\cdot(1+\sin\alpha)=0$.	2 points	
$\sin^2\alpha+\cos^2\alpha+2\sin\alpha\cos\alpha-1-\sin\alpha=0$	2 points	
Using the identity $\sin^2\alpha+\cos^2\alpha=1$ one gets $2\sin\alpha\cos\alpha-\sin\alpha=0$.	1 point	
First solution		
$\sin\alpha(2\cos\alpha-1)=0$	2 points	*These 2 points are due for any feasible method.*
This holds if and only if a) $\sin\alpha=0$, that is $\alpha=k\pi$ with $k\in\mathbf{Z}$;	1 point	*If the candidate gets false roots and they are, in fact, accepted or the system of the roots is deficient then at most 3 points may be given out of these 5.*
b) $2\cos\alpha-1=0$,	1 point	
yielding $\cos\alpha=\frac{1}{2}$, $\alpha=\frac{\pi}{3}+2n\pi$ with $n\in\mathbf{Z}$,	1 point	
or $\alpha=\frac{5\pi}{3}+2m\pi$ with $m\in\mathbf{Z}$.	1 point	
The steps are equivalent and thus the roots satisfy the equation.	1 point	
Second solution		
Squaring both sides of $2\sin\alpha\cos\alpha=\sin\alpha$ and collecting the terms one gets $3\sin^2\alpha(3-4\sin^2\alpha)=0$.	2 points	
$\sin^2\alpha=0\Leftrightarrow\alpha=n\pi, n\in Z$	1 point	
$\sin^2\alpha=\frac{3}{4}\Leftrightarrow\alpha=\pm\frac{\pi}{3}+2k\pi$ or $\alpha=\pm\frac{2\pi}{3}+2l\pi; k,l\in Z$	2 points	
Sorting out the false roots.	2 points	
Total:	**13 points**	

4. a)		
The total number of votes at the time of the announcement is $10500 \cdot 0.76 \cdot 0.9 = 7182$	1 point	
The number of spoiled ballots so far is $7182-(2014+2229+2805)=134$	1 point	
that is $\approx$ 1.9% of the votes.	1 point	
Total:	**3 points**	

4. b)		
$\frac{2014}{7182} \approx 0.2804$, that is Alchemist got the 28% of the votes; $\frac{2229}{7182} \approx 0.3104$, that is Owl got the 31% of the votes; $\frac{2805}{7182} \approx 0.3906$, that is Flute got the 39% of the votes processed so far.	1 point	
The corresponding central angles are as follows: A— 101º; O — 112º; F — 140º	1 point	
Spoiled ballots (1.9%) — 7º	1 point	
The sketch of the pie chart	1 point	*This 1 point should be given out even if the candidate has forgotten about the spoiled ballots.*
Total:	**4 points**	

4. c)		
There are $7182:9=798$ ballots still to be counted.	1 point	
If all of them would be valid and all of them would go to Alchemist then he would, in fact, win the elections by a total of 2812 votes.	2 points	
Total:	**3 points**	

4. d)		
If the percentage in question is denoted by x then Flute is to win for sure if $0.95 \cdot \frac{x}{100} > 0.05$ that is $x > \frac{5}{0.95} \approx 5.3$.	3 points	
Therefore, Flute will certainly win the elections if he is on the lead by at least 5.3 % after having counted 95% of the votes.	1 point	
Total:	**4 points**	

II.

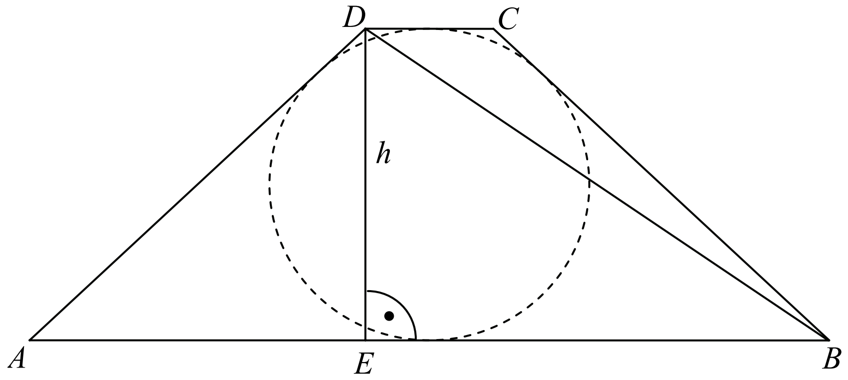
5. a)

Since Andrew and Bernie cover the route up in 20 minutes and 18.75 minutes, respectively, Andrew is moving downwards and Bernie is moving upwards at the instant of their encounter.	3 points	*These 3 poins should be given if the directions are deduced correctly from the graph. If the calculations are correct but the indication of the actual directions is missing then at most 2 points may be given.*
Denoting the distance, in kilometers, of the point of encounter from the peak by *x,* the total time, in hours used by Andrew is $\frac{1}{3}+\frac{x}{20}$,	2 points	
and the time, also in hours used by Bernie is $\frac{5-x}{16}$.	1 point	
Since Andrew parted 10 minutes earlier, one has $\frac{1}{3}+\frac{x}{20}=\frac{5-x}{16}+\frac{1}{6}$.	2 points	
Hence $x=\frac{35}{27}\text{ km}(\approx 1.3\text{ km})$.(This result satisfies the conditions.)	2 points	
Total:	**10 points**	*If the candidate ignores the condition that the runners parted at different moments then at most 5 points may be given for part a). If it is only the moment of the encounter that is calculated but not its distance from the peak then at most 8 points may be given.*

5. b)

The condition can be satisfied only if the girls were acquainted to 0, 1, 2, …, 9 boys, respectively.	2 points	
The total number of acquaintances between the boys and the girls is $1+2+3+...+9=45$.	2 points	
Therefore, if the 9 boys had known 6 of the girls, respectively, then the number of acquaintances would have been 54, a contradiction.	2 point	
Total:	**6 points**	*An incomplete but feasible attempt using graphs or some arithmetic argument may be given at most 3 points.*

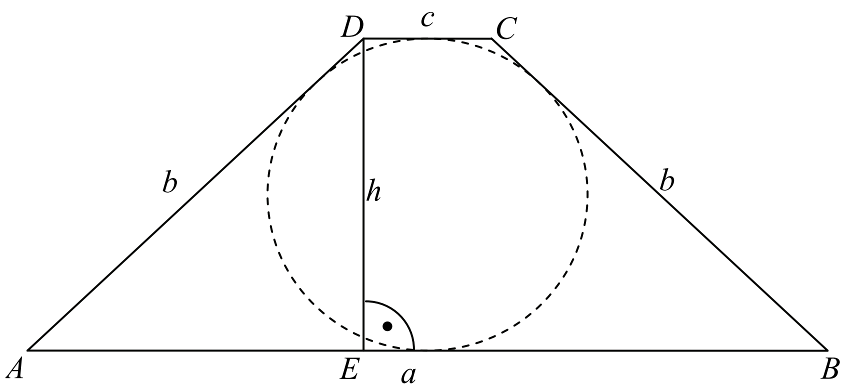
6. a)



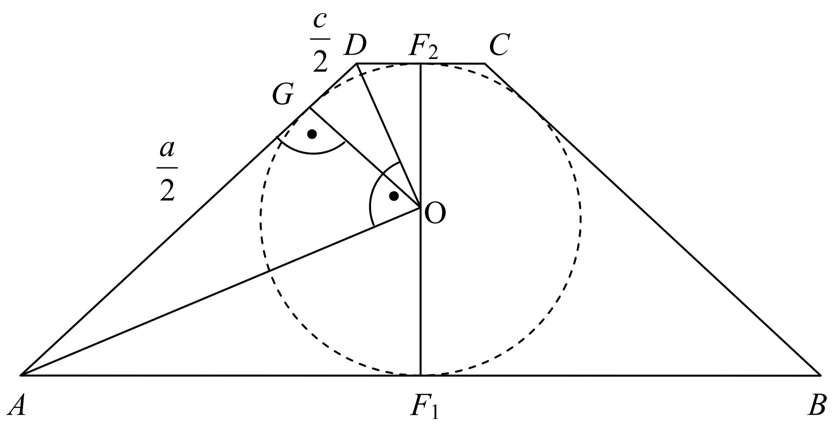
In the trapezium $ABCD$ one has $AB = 20$, $CD = 5$. Denote the foot of the altitude from D on the side AB by E. With these notations $AE = \frac{AB - CD}{2} = \frac{15}{2}$, and thus $EB = \frac{25}{2}$.	1 point	*These 3 points are still due if h is calculated by using the theorem in c) correctly.*
The given trapezium has an inscribed circle, therefore $AD = \frac{AB + CD}{2} = \frac{25}{2}$.	1 point	
By Pythagoras' theorem $h = DE = \sqrt{AD^2 - AE^2} = 10$.	1 point	
$Area = \frac{AB + CD}{2} \cdot h = 125$.	1 point	
In the triangle BED: $BD = \sqrt{DE^2 + EB^2} = \frac{5\sqrt{41}}{2} (\approx 16.01)$.	1 point	
Total:	**5 points**	

6. b)

The given solid of revolution consists of a cylinder and two congruent circular cones.	1 point	*This point is due if this idea appears in the solution.*
The radius of the common base circle of the cylinder and the cones is equal to $r = h = 10$.	1 point	
The height of the cylinder is $CD = 5$ and that of the cones is $AE = \frac{15}{2}$.	2 point	
The volume of the solid is hence $V = 2 \cdot \frac{r^2\pi \cdot AE}{3} + r^2\pi \cdot CD = 1000\pi (\approx 3141.59)$.	1 point	
Total:	**5 points**	

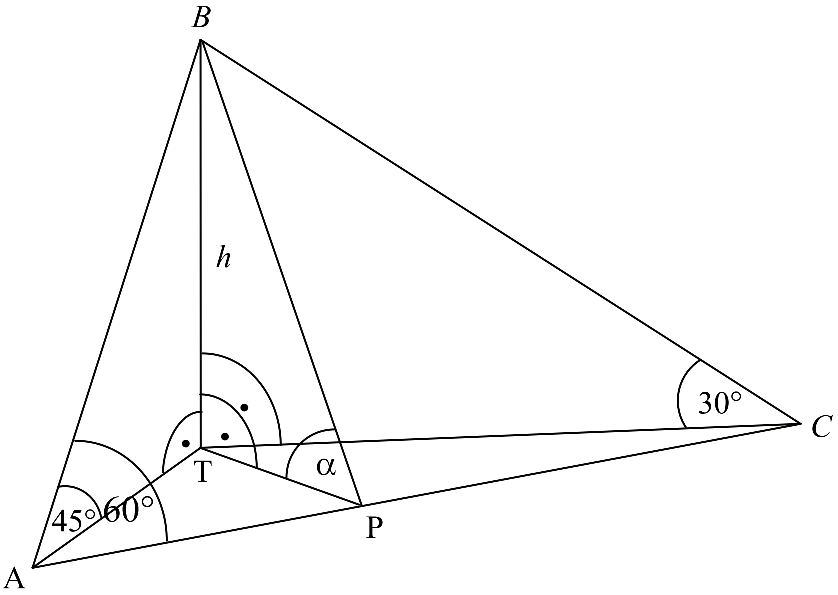
6. c) first solution		
		
Denoting the lengths of the bases of the given trapezium by a and c ($a \geq c$), the length of its edges by b, and its height by h the claim is $h = \sqrt{ac}$, or $h^2 = ac$.	1 point	
The trapezium has an inscribed circle, therefore $b = \frac{a+c}{2}$.	1 point	
Using the notations of the diagram $AE = \frac{a-c}{2}$ by symmetry..	1 point	
By Pythagoras' theorem $h^2 = \left(\frac{a+c}{2}\right)^2 - \left(\frac{a-c}{2}\right)^2 =$	1 point	
$= ac$.	2 point	
Total:	**6 points**	

6. c) second solution



Denote the incenter by O and the touching point of the incircle at the side AD by G. The tangents from an external point to a circle are equal an thus, using the notations of the diagram, $AG = AF_1 = \frac{a}{2}$ és $DG = DF_2 = \frac{c}{2}$.	1 point	*This point is due if this idea appears in the solution or it is apparent in the diagram.*
The sum of the angles lying on the edge of a trapezium is equal to 180°. Since O is the intersection of the internal angle bisectors one gets $DAO\angle + ODA\angle = 90^\circ$, that is the triangle AOD is right angled at O	1 point	
and its altitude perpendicular to the hypotenuse is the radius of the incircle. (OG),	1 point	
which is the half of the altitude of the trapezium.	1 point	
By the rule of height in the triangle AOD one gets	1 point	
$\frac{h}{2} = \sqrt{\frac{a}{2} \cdot \frac{c}{2}} \Leftrightarrow h = \sqrt{ac}$, and that was the claim to be proved.	1 point	
Total:	**6 points**	

7.



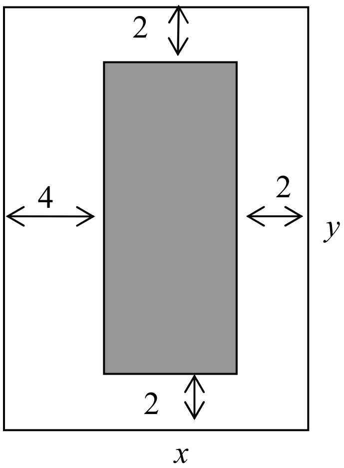
7. a)

ATB is an isosceles right triangle, therefore	1 point	
$AB = h\sqrt{2}\ (\approx 1191)$.	1 point	
In the right triangle *CBT* $BC = \dfrac{h}{\sin 30°} = 2h(\approx 1684)$.	2 points	
By the cosine rule, in the triangle *ACB* $CB^2 = AB^2 + AC^2 - 2 \cdot AB \cdot AC \cdot \cos 60°$,	1 point *	
$4h^2 = 2h^2 + AC^2 - \sqrt{2} \cdot h \cdot AC$,	1 point *	*These 2 points are due for the equation* $AC^2 - 1191AC - 1417375$ *=0.*
$AC^2 - \sqrt{2} \cdot h \cdot AC - 2h^2 = 0$.	1 point *	
Since it is positive, $AC = \dfrac{\sqrt{2}+\sqrt{10}}{2} h \approx 1927$ meter.	1 point *	
Total:	**8 points**	

** If the candidate calculates the missing angles in the triangle ACB first, then each of them is worth 1-1 points* $(ACB\angle = 37{,}76°; ABC\angle = 82{,}24°)$*; 1 more point for proceeding by the cosine rule correctly, finally 1 point for the correct result.*

7. b)		
In the right triangle *PBT* one gets $\tan\alpha = \frac{h}{TP}$.	2 point	
Since the angle α is acute and the tangent function is monotonically increasing in this interval, one can work with the fraction $\frac{h}{TP}$.	1 point	
The numerator is constant, therefore the fraction is maximal if *TP* is minimal.	1 point	
In fact, this is the case when *TP* is perpendicular to the line *AC* being the height of the triangle *ACT*. Therefore, the position of the maximum is indeed the foot of the height from *T* of the triangle *ACT*.	1 point	
Total:	**5 points**	

7. c)		
By the condition: $p_0 e^{Ch} = 0.8 p_0$.	1 point	
$h = \frac{\log_e 0.8}{C}\left(= \frac{\lg 0{,}8}{C \lg e}\right)$,	1 point	
The balloon arrived at the altitude $h \approx 1783\,(\mathrm{m})$.	1 point	
Total:	**3 points**	



8. a) first solution

Denote the dimensions of a page by x and y.	1 point	*This point is due if the meaning of the unknowns is clear from the diagram.*
$xy = 600$, $y = \frac{600}{x}$. (1)	1 point	
The printing area is $A = (x-6)(y-4)$, (2)	1 point	
with $x > 6$ and $y > 4$; (that is $x \in \,]6;150[$)	1 point *	
Plugging (1) into (2) yields $A(x) = (x-6)\left(\frac{600}{x} - 4\right)$.	1 point	
$A(x) = 624 - 4x - \frac{3600}{x}$	1 point	
We have to find the maximum of the function $A(x)$. $A'(x) = -4 + \frac{3600}{x^2}$	1 point	
$-4 + \frac{3600}{x^2} = 0$.	1 point	
The root of this equation lying in the domain is $x = 30$.	1 point	*The 1 point marked by * is also due if the candidate is clarifying the domain of the variable at this phase only.*
The second derivative is: $A''(x) = -2 \cdot \frac{3600}{x^3}$.	1 point	*These 2 points are due even if, instead of working with the second derivative, the candidate is properly checking the sign of the first derivative.*
Since the second derivative is negative for every positive value of the variable, and thus, in particular, at $x = 30$, the function A assumes its maximum here.	1 point	
The dimensions of the optimal page are 30 cm and 20 cm, respectively.	1 point	
Total:	**12 points**	

8. a) second solution		
Denote the dimensions of a page by x and y	1 point	*This point is due if the meaning of the unknowns is clear from the diagram.*
The typing area is $A=(x-6)(y-4)$,	1 point	
with $x > 6$ and $y > 4$.	1 point *	
$A=xy-4x-6y+24=624-2\cdot(3y+2x)$	1 point	
A is maximal if and only if $3y+2x$ is minimal.	1 point	
By the AM-GM inequality $\frac{3y+2x}{2}\ge\sqrt{3y\cdot 2x}$,	2 point	*The 1 point marked by * is also due if the candidate is clarifying the domain of the variable at this phase only.*
and, since xy=600 $\frac{3y+2x}{2}\ge\sqrt{6xy}=\sqrt{6\cdot 600}=60$.	2 points	
The minimum value 60 on the r.h.s. is assumed if and only if $3y=2x$.	1 point	
Hence, using xy=600 one gets $x = 30$ and $y = 20$.	1 point	
The dimensions of the optimal page are 30 cm and 20 cm, respectively.	1 point	
Total:	**12 points**	

8. b)		
The page numbers occuring on the printed pages are those from 3 to 122	1 point	*These 2 points are due no matter how does the candidate find the number of page numbers containing the digit 2. A correct answer is 1 point,and another 1for the reasoning.*
and there are 23 of these numbers containing the digit 2.	1 point	
The probability is hence $P=\frac{23}{120}(\approx 0.1917)$.	2 points	
Total:	**4 points**	

9. a)		
There are $\binom{9}{3}$ ways to choose the 3 chairs out of 9.	1 point	
There are $3!=6$ different orders of the professors to be seated on the selected 3 chairs..	1 point	
The total number of ways hence is $\binom{9}{3}\cdot 6 = 9\cdot 8\cdot 7 = 504.$	2 point	
Total:	**4 points**	*These 4 points are still due if the candidate writes down with explanation the correct answer as the number of certain variations. 2 points should be given for a correct answer without explanation.*

9. b)		
There are 5 seats in between the 6 students for the professors to it down. The 3 professors may choose among $\binom{5}{3}=10$ possibilities.	2 point	
There are $6!=720$ orders of the students among themselves and	1 point	
$3!=6$ orders of the professors tos sit down.	1 point	
Therefore, there are $\binom{5}{3}\cdot 6!\cdot 3! = 43200$ arrangements, altogether.	2 point	
Total:	**6 points**	

<table>
<tr><td colspan="3">9. c)</td></tr>
<tr><td>The medals can be handed out in $9!$ order.</td><td>1 point</td><td rowspan="3">If the candidate calculates the result as the product ($\frac{6}{9}\cdot\frac{1}{8}\cdot\frac{5}{7}=\frac{5}{84}$) of probabilities then the explanation of each factor is worth 1 point and the reference to the independence of the events involved is an addiotional point.</td></tr>
<tr><td>There are $6\cdot 5=30$ ways to choose the first one and the third one from the students.</td><td>2 point</td></tr>
<tr><td>Apart from the professor of Biology and the two students previously selected there are 6! orders for the remaining 6 to be called.</td><td>1 point</td></tr>
<tr><td>Hence the probability in question is
$P=\frac{30\cdot 6!}{9!}=\frac{5}{84}\approx 0{,}06$.</td><td>2 point</td><td></td></tr>
<tr><td align="right">Total:</td><td>6 points</td><td></td></tr>
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